Functions

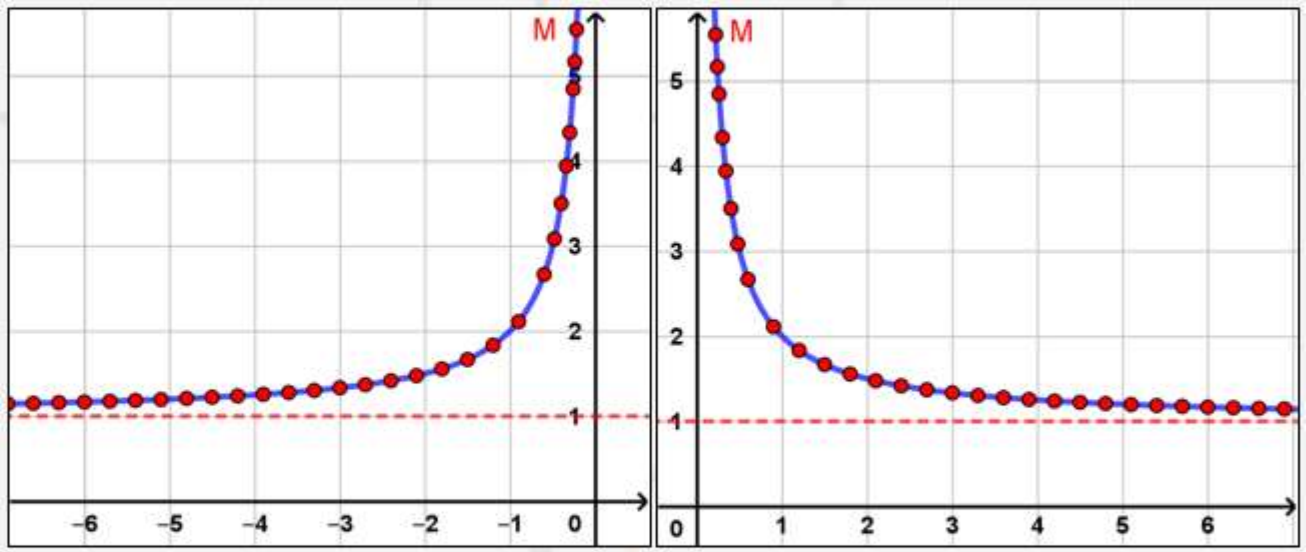
Asymptotes

Horizontal asymptote

As x increases indefinitely to +∞ or decreases indefinitely to -∞, $f(x)$ can approaches toward a horizontal line of equation $y = b$ i.e. $\lim_{x \to \pm\infty} f(x) = b$

This line is called:

horizontal asymptote



Horizontal asymptote

Application # 1

Consider the function f defined over IR by $f(x) = \frac{-4x^2+3x-1}{2x^2+1}$.

Show that the line of equation $y = -2$ is a horizontal asymptote at $\pm\infty$.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{-4x^2}{2x^2} = -2$$

so $y = -2$ is a horizontal asymptote at $\pm\infty$

Horizontal asymptote

Application # 2

Consider the function f defined over IR by $f(x) = \begin{cases} \frac{x^3}{x^3+1} \; if \; x \geq 0 \\ \frac{1}{x} \; if \; x < 0 \end{cases}$.

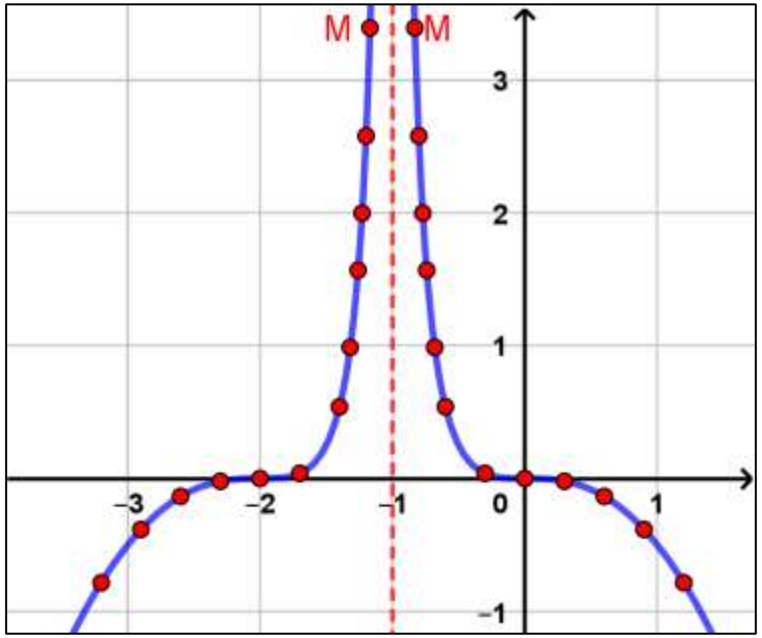
Study the existence of horizontal asymptotes at $\pm\infty$.

$\lim_{x \to +\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^3}{x^3} = 1$ so $y = 1$ is a horizontal asymptote at $+\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$ so $y = 0$ is a horizontal asymptote at $-\infty$

Vertical asymptote

If a point M of the curve moves away indefinitely while approaching toward a vertical line of equation $x = a$, this line is called **vertical asymptote** of the the curve.



In this case:

$$\lim_{x \to a} f(x) = \pm\infty$$

Remark: the function f is not defined at x=a.

Vertical asymptote

Application # 1

Consider the function f defined by $f(x) = \frac{2x}{x+1}$.

Show that the line of equation $x = -1$is a vertical asymptote at $\pm\infty$.

f is defined when $x \neq -1$

$$\lim_{x\to -1} f(x) = \frac{2(-1)}{-1+1} = -\frac{2}{0} = \begin{cases} \lim\limits_{x\to -1^-} f(x) = -\frac{2}{0^-} = +\infty \\ \lim\limits_{x\to -1^+} f(x) = -\frac{2}{0^+} = -\infty \end{cases}$$

So $x = -1$ is a vertical asymptote.

Vertical asymptote

Application # 2

Consider the function f defined by $f(x) = \frac{x^2-9}{x-3}$.

Show that the function f doesn't have any vertical asymptote.

f is defined when $x \neq 3$

$$\lim_{x \to 3} f(x) = \frac{0}{0} \text{ I.F.}$$

$$\lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} x + 3 = 3 + 3 = 6$$

so $x = 3$ is not a vertical asymptote.

Oblique asymptote

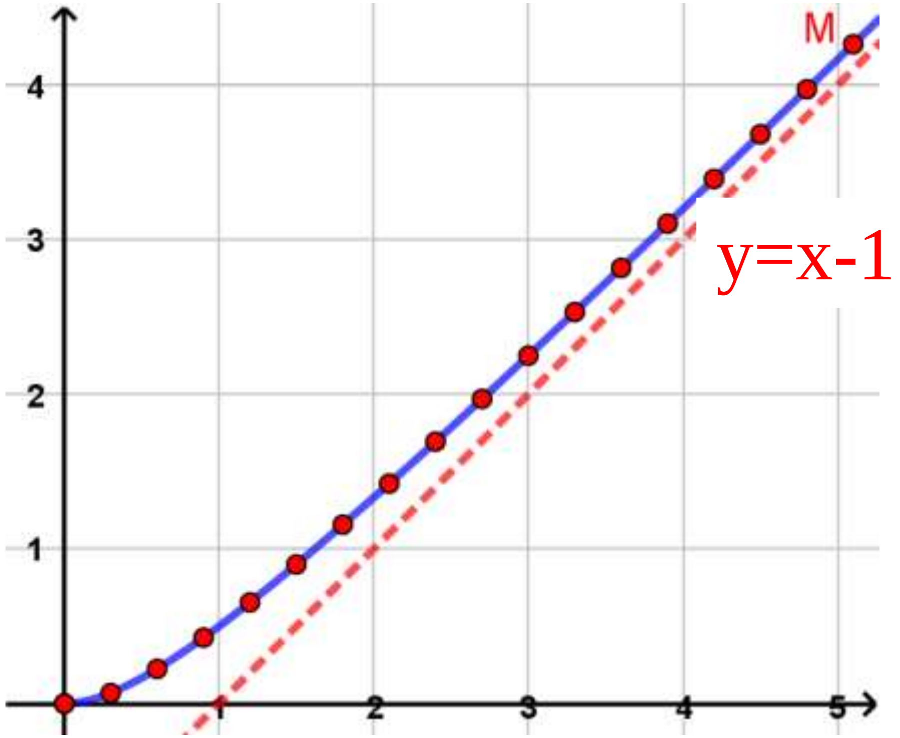
If a point M of the curve moves away indefinitely while approaching toward an oblique line of equation y=ax+b, this line is called **oblique asymptote** of the curve.

In this case:

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

And

$$\lim_{x \to \pm\infty} (f(x) - y) = 0$$



Oblique asymptote

Application # 1

Show that the line (d) of equation $y = x$ is an oblique asymptote of the function f defined by $f(x) = \frac{x^3+1}{x^2}$.

$$f(x) - y = \frac{x^3+1}{x^2} - x = \frac{x^3+1-x^3}{x^2} = \frac{1}{x^2}$$

$$\lim_{x\to\pm\infty}(f(x) - y) = \lim_{x\to\pm\infty}\frac{1}{x^2} = \frac{1}{+\infty} = 0$$

So (d) is an oblique asymptote at $\pm\infty$.

Oblique asymptote

Application # 2

Consider the function f defined by $f(x) = \frac{x^2+4x-1}{x}$.

Show that the line (d) of equation $y = x + 4$ is an oblique asymptote.

$$f(x) - y = \frac{x^2+4x-1}{x} - (x+4) = \frac{x^2+4x-1-x^2-4x}{x} = -\frac{1}{x}$$

$$\lim_{x\to\pm\infty}(f(x) - y) = \lim_{x\to\pm\infty}\frac{-1}{x} = \frac{-1}{\pm\infty} = 0$$

So (d) is an oblique asymptote at $\pm\infty$.

Application #3

Determine the equation of the asymptotes in each case.

Vertical asymptote: $x = 0$

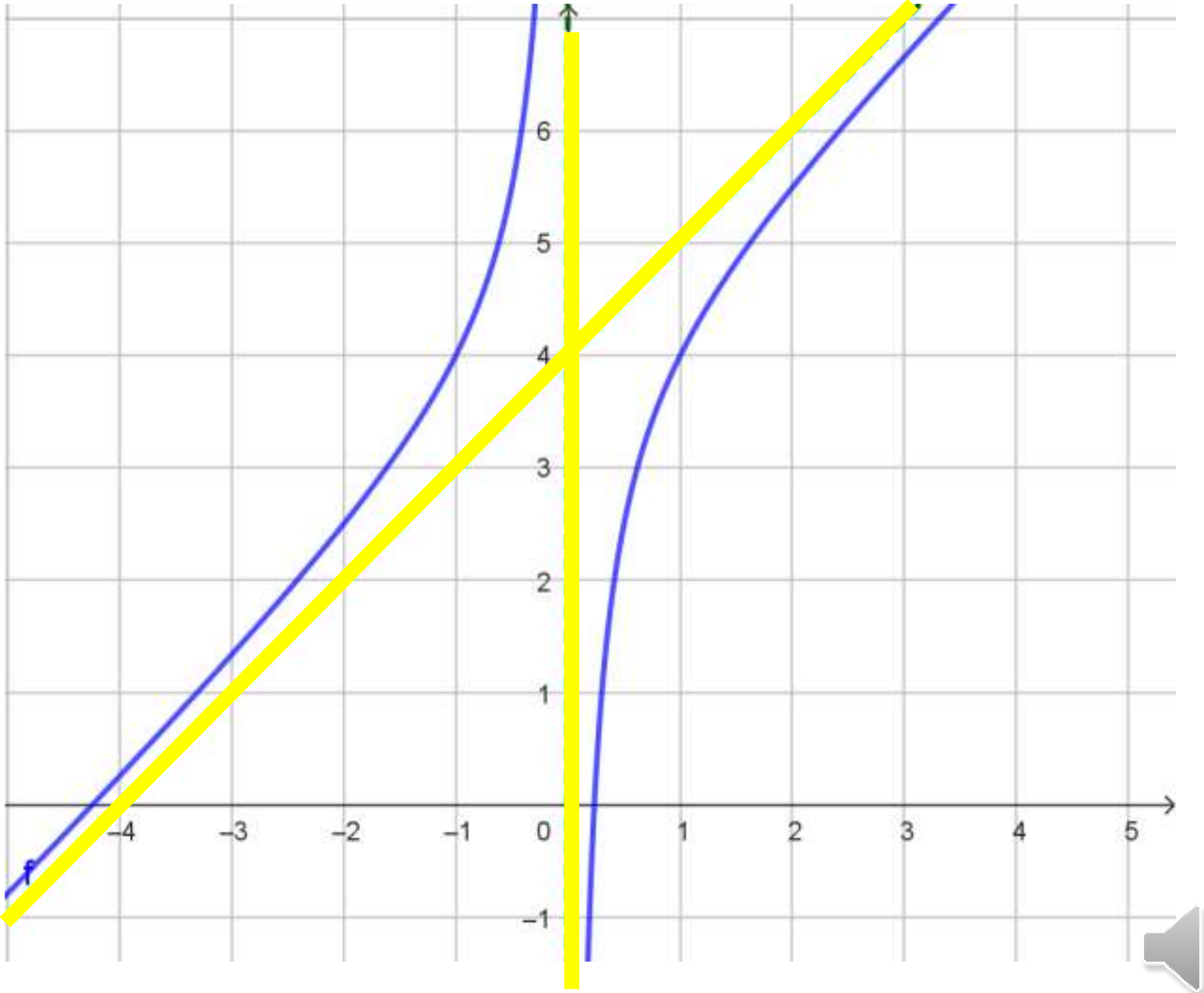
Oblique asymptote:
The line passes through the points A(0;4) and B(1;5)
The equation is :

$y - y_A = a(x - x_A)$

$a = \frac{y_A - y_B}{x_A - x_B} = \frac{4-5}{0-1} = 1$

$y - 4 = 1(x - 0)$

So the equation is $y = x + 4$



Application #3

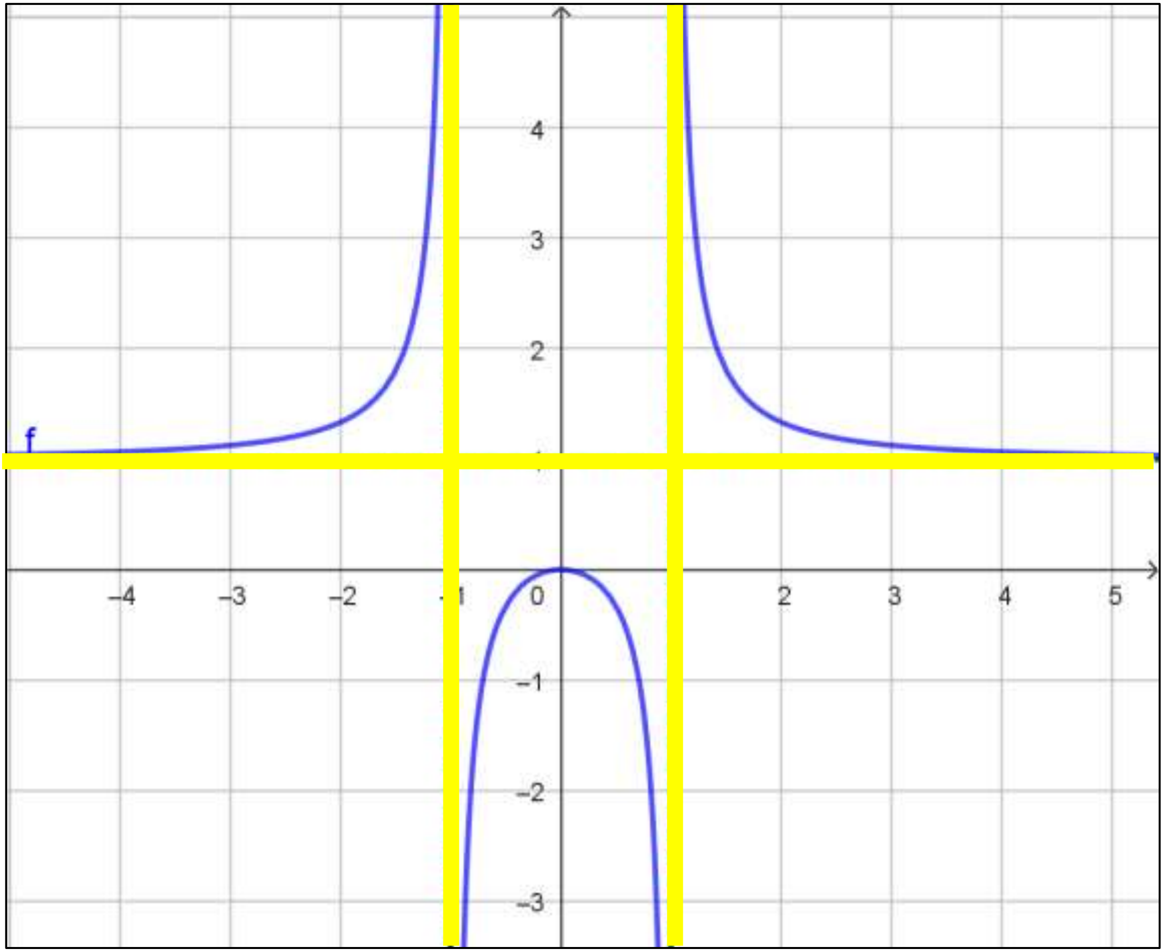
Determine the equation of the asymptotes in each case.

Vertical asymptote:

$x = -1$ and $x = 1$

Horizontal asymptote:

$y = 1$



Application #4

Which figure represent an asymptote?